The Melbourne University Mathematics and Statistics Society

presents

2005 University Maths Olympics

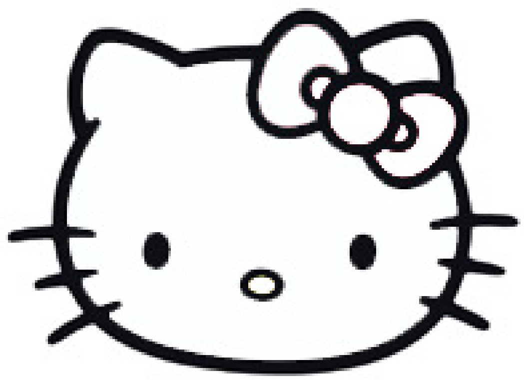
The Solutions

These are the solutions to the questions on the MUMS University Maths Olympics held on the 14th of September 2005 (umo05questions.pdf). It addresses some of the flaws in the questions and also contains a commentary on how the questions were generally handled during the competition.

Solutions prepared by Daniel Yeow.

Question 1

What is the minimum number of times a pen must be lifted off the page in order to draw the following diagram?



Solution to Question 1

The first thing to realise is that no mention is made of restrictions on going over lines already drawn. Many people made this mistake and guessed numbers which were far too high. In fact, the top two teams both skipped the question in frustration. The answer is, of course, 3.

Question 2

All smoons are goons, but only some goons are spoons. At least one smoon is little but no spoons are little. If all spoons dislike the goons who are little, and some goons who are not smoons are little but dislike spoons...

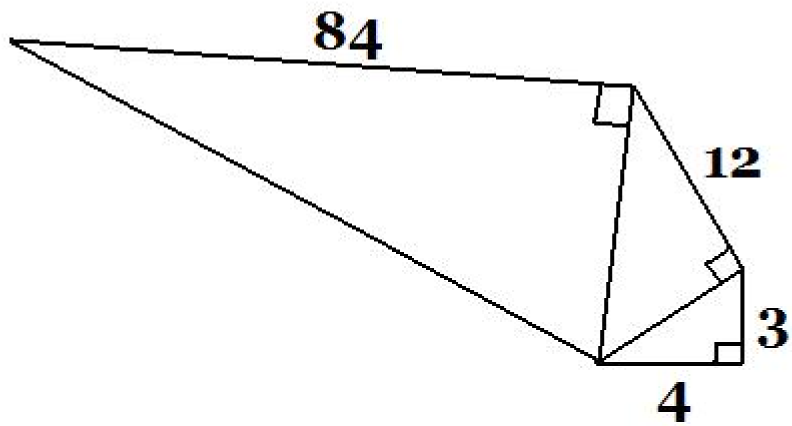
How many instances of double letters appear on this piece of paper?

Solution to Question 2

This question was met with many groans. I was told of a particularly amusing incident in which my Topology lecturer began drawing Venn diagrams on reading this question. It is simply a matter of counting double letters - time consuming and not particularly mathematically elegant, but a safety measure to reduce the probability of teams' runners from colliding.

Question 3

What is the length of the hypotenuse of the largest triangle?



Solution to Question 3

The first two pythagorean triples are well known (3,4,5), (5,12,13), the third less so. It can easily be found that $13^2 + 84^2 = 7225 = 85^2$. The answer is 85.

Question 4

Andrew is only useless at the Maths Olympics on any day whose name, when spelt in English, contains the letter 't' in it. (forget for the moment about yesterday, today and tomorrow). Nick is only useless on days with an 'i' in them. Geoff is only useless on a Sunday, James is only useless on Monday and Daniel is useless on all the days ending with 'y'. Sally is useless on all the days that Daniel is useless on but on which no-one else is useless on. On what day is Sally useless at the Maths Olympics?

Solution to Question 4

Writing out the days of the week and eliminating them one by one yields the answer fairly quickly. Realizing that the day of the UMO this year is, in fact, Wednesday and knowing something of the organisers' twisted sense of humour would lead a savvy olympian to guess 'Wednesday' straight away - which is the correct answer.

Question 5

In a rabbit race, the rabbit who came three places in front of the rabbit who finished last came two places ahead of the rabbit who came seventh. How many rabbits were in the race?

Solution to Question 5

There are two valid approaches to this problem, given UMO conditions. First, one can realise that 7 is a lower bound and begin guessing from 7 onwards. Alternatively, one can actually do the calculation, which is fairly trivial - two places ahead of 7th is 5th, 5th plus 3 places - 8 rabbits.

Question 6

How many powers of 37 are in the number $1^1 \times 2^2 \times \ldots \times 2005^{2005}$?

Solution to Question 6

This is the first non-trivial question of the UMO. Many got bogged down here, most teams eventually figured out the method, but eventually gave up due to frustration arising from errors in their calculations (which were, to be fair, done under great pressure).

Thinking about this, we realise that 37^{37} gives us 37 powers, $(2 \times 37)^{(2\times 37)}$ gives us 74 powers of 37, etc. Next we figure out the highest multiple of 37 which appears before 2005. This turns out to be $37 \times 54 = 1998$. So, we need to find out the sum of the series $37 + 74 + 111 + \ldots + 1998$, which simplifies to $(1 + 2 + 3 + \ldots + 54) \times 37$. The sum of the numbers from 1 to 54 is $\frac{54\times 55}{2} = 1485$ (which was submitted as the answer a few times). We then multiply this by 37 to get 54945.

But we are not done yet. $(37 \times 37)^{(37\times 37)}$ was only counted once and we need to count it again. 37^2 is 1369, so we just add it to 54945 to get 56314.

Question 7

How many squares are there on a chessboard?

Solution to Question 7

This one is fairly standard. There are 64 of those little squares. Then there are 49 of those 2×2 squares because counting them is equivalent to reducing the size of the board to a 7×7 board. So all one needs to do is sum 64, 49, 36, 25, 16, 9, 4, 1 and this comes to 204.

Question 8

A circular island of radius 100m sits in the exact centre of a regular hexagonal lake of side length 200m. The Queen of the island, being somewhat strapped for cash after purchasing an expensive map of France, needs to build a bridge because she can't swim. What is the minimum length of this bridge, in metres?

Solution to Question 8

First divide the hexagon into equilateral triangles, then find the length of any altitude - $100\sqrt{3}$. Then subtract the radius of the island to get $100\sqrt{3} - 100$ or $100(\sqrt{3} - 1)$.

Question 9

The year 2005 has reflective symetry when viewed in digital digits. How many years ago was the last year to have this property?

Solution to Question 9

Digits on a digital display (i.e. a digital alarm clock) is what we were thinking of. Not sure why, but many teams tried solving this in binary for some reason. The last year with this property was 1881, which was submitted as an answer by most teams. If one reads the questions carefully, one finds that we are actually asking how long it has been since that year, which is 124 years.

Question 10

'Twas last Bank Holiday, so I've been told,
Some cyclists rode abroad in glorious weather.
Resting at noon within a tavern old,
They all agreed to have a feast together.
'Put it all in one bill, mine host,' they said,
'For every man an equal share will pay.'
The bill was promptly on the table laid,
And four pounds was the reckoning that day.
But, sad to state, when they prepared to square,
'Twas found that two had sneaked outside and fled.
So, for two shillings more than his due share
Each honest man who had remained was bled.
They settled later with those rogues, no doubt.
How many were they when they first set out?
(note: there are 20 shillings in a pound)

Solution to Question 10

Suppose x cyclists started out. The bill was 80 shillings, so $\frac{80}{x} + 2 = \frac{80}{x-2}$. Hence $x^2 - 2x - 80 = 0$. This factorises as $(x - 10)(x + 8)$ so the answer is 10 as it must be positive.

Question 11

What is the next number in the following sequence?

$$1, 4, 9, 61, 52, \ldots$$

Solution to Question 10

In some languages, digits are written from right to left... writing the digits of this sequence reversed yields 1, 4, 9, 16, 25. This question is a little dodgy in the eyes of some because the solution is probably not strictly unique. The answer is, of course, 63.

Question 12

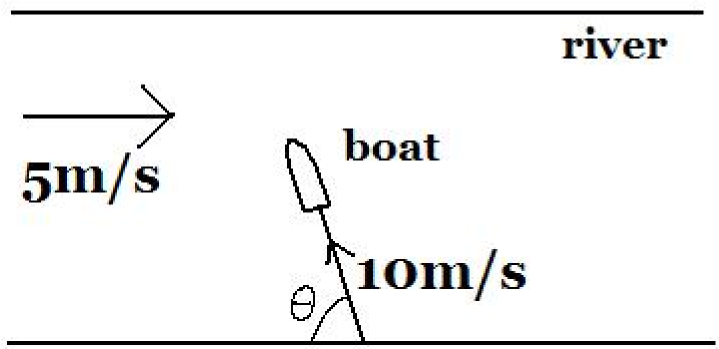
The game of Go is played on a 18×18 board composed of rectangles which are 12mm $\times$ 13mm. How many squares are there on a Go board?

Solution to Question 12

With 12×13 rectangles, to construct a square, one must select 13×12 of them. How many different places can one draw a 13×12 square on a Go board? Initially, I thought that since that leaves 5 rectangles on one side and 6 on its perpendicular counterpart, that the answer would be 30. However, on closer inspection, because the edges are included, the answer is actually 6×7 which is the meaning of life, the universe and everything - 42.

Question 13

A river with perfectly straight and parallel banks flows west to east at 5 metres per second. You have a badly-drawn boat which can only travel in straight lines and can go exactly 10 metres per second. In order to reach the north bank in the shortest amount of time, what is the angle in degrees, relative to the south bank that must the boat take (see diagram)?



Solution to Question 13

The point of this question was to trick people into calculating the path of least distance when, in fact, the question asks for the path of least time. The answer to the question is, of course, to just cast the boat off pointing straight northwards – 90^o.

Question 14

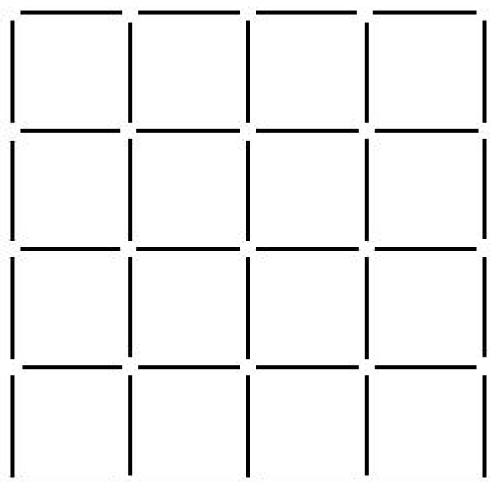
An equilateral triangle of side length n is divided into n^2 equilateral triangles of side length 1 by lines parallel to its sides, thus giving a network of nodes connected by line segments of length 1. What is the maximum number of segments that can be chosen so that no three chosen segments form a triangle?

Solution to Question 14

Every segment belongs to just one other of the $\frac{n(n+1)}{2}$ triangles with base horizontal. We can choose at most 2 sides of each of these triangles, or $n(n+1)$ in all. If we choose all the segments that are not horizontal, then we choose $n(n+1)$ segments. Since every triangle has one horizontal segment, no three chosen segments form a triangle.

Question 15

Given a 4×4 square grid made up of matchsticks (i.e. 40 matchsticks), what is the minimum number of matchsticks you can remove such that there are no squares left?



Solution to Question 15

I have two approaches to this question, neither of which are particularly elegant, but both will give the correct answer in a short amount of time.

First, begin by considering a 1×1 square, only one matchstick need be removed to 'break' all the squares. Then go to a 2×2 square, by inspection one can easily see that three matchsticks must be removed. Now try a 3×3 square. With a little fiddling, we find that at least six matchsticks must be removed. At this point I would guess the next number in the sequence - nine, which turns out to be the right answer.

Alternatively, we notice that there are 16 small squares meaning that there are 8 small squares which do not share an edge and thus 8 is a lower bound for the number of matchsticks. Also observe that if we take 8 matchsticks away to 'break' all the small squares, none of these matchsticks lie on the edge of the large square, which implies that at least one more matchstick must be removed giving a lower bound of 9. A good starting point for guessing.

Question 16

Given a regular tetrahedron (a pyramid whose faces are all equilateral triangles) of side length 2, how far from its centre is one of its vertices?

Solution to Question 16

This question was the stumbling block for many many teams. It was originally far earlier in the order until someone pointed out to me that it was, in fact, quite difficult if you did not immediately see the 'trick'. The trick is to place the tetrahedron into a cube. It is trivial to calculate that the ratio of the sidelength of a unit cube to the distance from one of its vertices to its centre is $1{:}\frac{\sqrt{3}}{2}$. The edges of a tetrahedron form the diagonals across the faces of the cube and the ratio of that length to the length from a vertex to the centre is $\sqrt{2} : \frac{\sqrt{3}}{2}$. We now expand our cube by a factor of $\sqrt{2}$ so that our tetrahedron fits. The answer is thus $\frac{\sqrt{3}\sqrt{2}}{2}$ or $\frac{\sqrt{3}}{\sqrt{2}}$.

Question 17

An equilateral triangle with side-length $22\sqrt{3}$ contains two circles which are tangent to each other and are each tangent to two sides of the triangle. The radii of these circles differs by 2. What is the sum of their radii?

Solution to Question 17

This question has particular sentimental value to myself because it is basically a reworking of a question from the first ever Schools Maths Olympcis (SMO) in which my team placed a very close second to another school team. This was the question which we were stuck on and had we solved it, we would have won. As an aside, most of the members from those two teams now make up most of the MUMS committee.

Drawing a diagram is the first thing to do. Taking the radius of the smaller circle to be n one obtains, from various fairly trivial trigonometric identities and a bit of pythagoras the equation:

$$n\sqrt{3} + 2\sqrt{n(n+2)} + (n+2)\sqrt{3} = 22\sqrt{3}$$

One could rearrange this to make a quadratic equation and then solve it, but looking at the equation and letting the 'zen' take over, one sees that $n = 6$ obviously works. $6 + 8 = 14$ which is the answer.

Question 18

How many ordered 4-tuples (a, b, c, d) are there such that $a + b + c + d = 25$?
(note: $a + b + c + d = 25$ and $a + b + d + c = 25$ where $c \neq d$ are *different* ways and $a, b, c, d \in \mathbb{N}$).

Solution to Question 18

In my seminar earlier this year on beauty and zen in mathematics, I went through the solution to a very similar problem. The trick is to consider the numbers from one to 25 as 25 objects arranged in a straight line. The '+' signs are merely dividers along the line. There are 3 dividers and 24 possible spots for the dividers to go, so the answer is clearly $\binom{24}{3} = \frac{24!}{21! \times 3!} = 2024$.

Question 19

A number n has sum of digits 100, whilst $44n$ has sum of digits 800. Find the sum of the digits of $3n$.

Solution to Question 19

Suppose n has digits $a_1, a_2, \ldots a_k$ and digit sum s $= \Sigma a_i$. If we temporarily allow digits larger than 9, then $4n = (4a_1)(4a_2)\ldots(4a_k)$. Each carry of one reduces the digit sum by 9, so after making all necessary carries, the digit sum for $4n$ is at most $4s$. It can only be $4s$ if and only if there are no carries. Similarly, $11n$ has digit sum at most $2s$, with equality if and only if there are no carries.

Since $44n$ has digit sum $8s$, there cannot be any carries. In particular, there are no carries in forming $4n$, so each digit of n is at most 2. Hence there are no carries in forming $3n$ and the digit sum of $3n$ is 300.

Question 20

$1! = 1$, $2! = 2$, $4! + 0! + 5! + 8! + 5! = 40585$. Find all other numbers that possess this property.

Solution to Question 20

Luckily, there is only one other number possessing this property. In any case, good UMO strategy dictates that one should guess any answers as soon as one has a reasonable answer. Luckily, the number is also a fairly low one. I almost changed the question to give the three lower numbers and ask the competitors to figure out the higher one, but it was decided in the end that that would have been too difficult even for the last question.

First one realises that factorials with more than two digits (i.e. $10!$) are out which leaves 1-9 to play with. It is trivial to show that there are no more one digit numbers with this property. Considering two digit numbers, we realise that $4! = 24$ and, since $5! = 120$ we can now also rule out two digit numbers. Notice also that the larger number dominates the end result. Begin with a number, say 5, and add combinations of the smaller factorials to see what numbers you get, remembering that both $0!$ and $1!$ are equal to 1. We quickly find that $5! + 4! + 1! = 145$ and, since order of addition doesn't matter, we can write $1! + 4! + 5! = 145$ and we are done.

This question is only easy because the answer is exceedingly easy to stumble upon in the 'intial playing' stage of going about solving it. My solution is essentially a case bash, but a quick one.

Question 21

ABC is a triangle with area 1. AH is an altitude, M is the midpoint of BC and K is the point where the angle bisector at A meets the segment BC. The area of the triangle AHM is $\frac{1}{4}$ and the area of AKM is $1 - \frac{\sqrt{3}}{2}$. Find the angles of the triangle in degrees.

Solution to Question 21

Assume $AB \geq AC$. Then $\frac{BK}{KC} = \frac{AB}{AC} \geq 1$, so K lies between M and C. Put $MH = x$. Then since area $AHM = \frac{1}{4}$ area ABC, we have $BC = 4x$, and hence $BM = 2x$, $HC = x$. Since the area $AKM = 1 - \frac{\sqrt{3}}{2}$, we have $MK = (4 - 2\sqrt{3})x$. Hence $\frac{AB}{AC} = \frac{BM+MK}{MC-MK} = \frac{6-2\sqrt{3}}{2\sqrt{3}-2} = \sqrt{3}$.

We have $AB^2 - BH^2 = AH^2 = AC^2 - CH^2$, so $AB^2 - AC^2 = (3x)^2 - x^2$. Hence $AC^2 = 4x^2$. So $AC : AB : BC = 1 : \sqrt{3} : 2$. Hence $A = 90^o$, $B = 30^o$, $C = 60^o$.

Or you could guess it...

Question 22

Find all solutions (x, y) in positive integers to $x^3 - y^3 = xy + 61$.

Solution to Questions 22

Put $x = y + a$. Then $(3a - 1)y^2 + a(3a - 1)y + (a^3 - 61) = 0$. The first two terms are positive, so the last term must be negative, so $a = 1, 2, 3$. Trying each case in turn, we get $(y + 6)(y - 5) = 0$, $5y^2 + 10y - 53 = 0$, $4y^2 + 12y - 17 = 0$. The last two equations have no integer solutions. The answer is $(6, 5)$. Alternatively, you might notice that $61 = 6^2 + 5^2$ and, given that UMO questions are often best solved on hunches, one could guess the answer.

Question 23

What is the minimum real value of $|\sin(x) + \cos(x) + \tan(x) + \cot(x) + \sec(x) + \operatorname{cosec}(x)|$?

Solution to Question 23

Put $x = y - \frac{3\pi}{4}$. Then $\sin(x) = \frac{-(\cos(y)+\sin(y))}{\sqrt{2}}$ and $\cos(x) = \frac{-(\cos(y)-\sin(y))}{\sqrt{2}}$, so $\sin(x) + \cos(x) = -\sqrt{2}\cos(y)$. After a bit of manipulation, $\tan(x) + \cot(x) = \frac{2}{\cos^2(y)-\sin^2(y)}$, $\sec(x) + \operatorname{cosec}(x) = \frac{-2\sqrt{2}\cos(y)}{\cos^2(y)-\sin^2(y)}$. After some further manipulation $\sin(x) + \cos(x) + \tan(x) + \cot(x) + \sec(x) + \operatorname{cosec}(x) = -c - \frac{2}{c+1}$, where $c = \sqrt{2}\cos(y)$. By the AM/GM inequality, $(c + 1) + \frac{2}{c+1}$ has minimum $2\sqrt{2}$ for $c + 1$ positive, so the expression has a minimum of $2\sqrt{2} - 1$ as required.

Question 24

There are 15 sea urchins sitting in 3 rows of 5. The sea urchins leave one at a time. All leaving orders are equally likely. Find the probability that there are never two rows where the number of sea urchins remaining differs by 2 or more.

Solution to Question 24

This was, in my opinion, the most difficult question on the entire UMO because it was among the most difficult to guess and it was also one of the most difficult (for me anyway) to do.

Initially, the question was written with 3 rows of n sea urchins. A number was decided on because it would be easier to mark, and it was decided that nothing was going to be gained (or lessened) in difficulty by having n. The solution is for 3 rows of n, then we simply substitute 5 into n.

Initially, we have n, n, n. It does not matter which sea urchins leave first. We get n, n, $n-1$ (in whatever order). Now one of the sea urchins in the fuller rows must leave, which has the probability $\frac{2n}{3n-1}$, giving n, $n-1$, $n-1$ (in whatever order). Now one of the sea urchins in the fullest row must leave, which has the probability $\frac{n}{3n-2}$ and gives $n-1$, $n-1$, $n-1$. Thus we have probability of $\frac{6n^3}{3n(3n-1)(3n-2)}$ of getting from n, n, n to $n-1$, $n-1$, $n-1$. Continuing in this manner, we have the probability of $\frac{6^n(n!)^3}{(3n)!}$ of completing the job. Substituting $n = 5$ into the equation (the computation is left as an excersise for the reader) gives the somewhat aesthetically pleasing fraction of $\frac{72}{7007}$.

Question 25

Given positive reals a, b, c find all real solutions (x, y, z) to the equations $ax+by = (x-y)^2$, $by + cz = (y - z)^2$, $cz + ax = (z - x)^2$.

Solution to Question 25

This question, despite being somewhat fiendish, is actually relatively easy to guess. No one did... But another quirk of question 25 on this particular UMO is that it was worth 31 points while questions 21-24 were worth 30 points. This was done so that, if two very talented teams finished all the questions bar the last two, if one team solved 24 while the other skipped 24 and correctly guessed 25, then they would be rewarded with a 1 point victory for taking the risk. As it turned out, I grossly overestimated the ability of the teams as the scoreline suggest. The winning score was 340 points out of a total possible 501 points - not even good enough for a H2B.

As for the solution... we have
$2ax = (ax + by) - (by + cz) + (cz + ax) = 2(x^2 - xy + yz - xz) = 2(y - x)(z - x)$
$2by = (ax + by) + (by + cz) - (cz + ax) = 2(y^2 + xz - zy - yz) = (z - y)(x - y)$
$2cz = -(ax + by) + (by + cz) + (cz + ax) = 2(z^2 + xy - yz - zx) = 2(x - z)(y - z)$

Now suppose $z \geq x, y$. Then if $x \geq y$, $ax = (z-x)(y-z) \geq 0$, and $by = (z-y)(x-y) \geq 0$. But a and b are positive, so $x \leq 0$ and $y \geq 0$. Hence $x = y = 0$. On the other hand, if $x \leq y$, then $ax \geq 0$ and $by \leq 0$, so again $x = y = 0$. Hence $cz = z^2$, so $z = 0$ or c. This gives the two solutions $(x, y, z) = (0, 0, 0)$ and $(0, 0, c)$.

Similarly, if $y \geq x, z$ then we find $x = z = 0$ and hence $y = 0$ or b. If $x \geq y, z$ then we find $y = z = 0$ and $x = 0$ or a. The answers are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.